Generalized Friedland-Tverberg inequality: applications and extensions

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Abstract

We derive here the Friedland-Tverberg inequality for positive hyperbolic polynomials. This inequality is applied to give lower bounds for the number of matchings in r-regular bipartite graphs. It is shown that some of these bounds are asymptotically sharp. We improve the known lower bound for the three dimensional monomer-dimer entropy. We present Ryser-like formulas for computations of matchings in bipartite and general graphs. Additional algorithmic applications are given.

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1 Introduction

The aim of this paper is to explore the connections between the problem of counting the number of partial matchings in graphs and positive hyperbolic polynomials, and Ryser-like formulas for partial matchings. Given a graph $G = (V, E)$ on N vertices, i.e. $\#V = N$, we want to compute the number of m-matching, i.e. the number of subsets M of edges E, where $\#M = m$, and no two edges in M have a common vertex.

Our main results are for bipartite graphs $G := (V_1 \cup V_2, E)$, where $E \subset V_1 \times V_2$ and $n = \#V_1 = \#V_2$. Let $A(G) \in \{0,1\}^{n\times n}$ be the incidence matrix of the bipartite graph G. Then the number of m-matchings in G is equal to $\mathrm{perm}_m A(G)$, where $\mathrm{perm}_m A$ is the sum of $m \times m$ minors of $A \in \mathbb{R}^{n\times n}$. For $m = n$, $\mathrm{perm}\, A(G)$, the permanent of $A(G)$ is the number of perfect matchings in G.

It is well known that the computation of the number of perfect matching in a general bipartite graph is $\#P - complete$. See [32] for the first proof and [3] for a simplified proof.

Ryser's algorithm to compute the permanent of any $A \in \mathbb{R}^{n\times n}$ [28] remains the most efficient exact algorithm, even though it uses around $n2^n$ operations. One can speed up significantly the approximate computation of $\mathrm{perm}\, A$. One knows to compute the permanent of a nonnegative matrix $A = [a_{ij}] \in \mathbb{R}_+^{n\times n}$ within a simply exponential factor [2]. In the case that all the entries of the matrix are uniformly bounded below and above by positive constants, one can improve the estimates of the exponential errors [26] and [15]. A fully randomized polynomial approximation scheme *frpas* for the number of perfect matchings in a bipartite G, and more generally for the permanent of a nonnegative matrix is given in

[25]. This result is generalized in [12] to the number of m-matchings in bipartite G, and more generally to $\mathrm{perm}_m A$, for any $A \in \mathbb{R}_+^{l\times n}$.

We now describe our main results for $\mathrm{perm}_m A$, where A is doubly stochastic, and their applications to lower bounds on partial matchings in bipartite graphs. Recall that the minimum of the permanent of $n\times n$ doubly stochastic matrices, denoted by Ω_n, is achieved only for the flat matrix J_n, whose all entries equal to $\frac{1}{n}$. Thus $\mathrm{perm}\, B \ge \mathrm{perm}\, J_n = \frac{n!}{n^n}$ for any $B \in \Omega_n$ and this inequality was conjectured by van der Waerden [33]. This conjecture was independently proved by Egorichev [5] and Falikman [7]. We call the above inequality Egorichev-Falikman-van der Waerden (EFW) inequality. The asymptotic behavior of EFW inequaity is captured by the inequality $\mathrm{perm}\, B \ge e^{-n}$ for any $B \in \Omega_n$. This inequality was shown by the first name author [8] three years before [5, 7]. Let $\Gamma(n,r)$ be the set of all r-regular bipartite graphs G on $2n$ vertices. For $G \in \Gamma(n,r)$ the matrix $B := \frac{1}{r}A(G)$ is doubly stochastic. Hence the number of perfect matchings in G is at least $(\frac{r}{e})^n$. Thus for $r \ge 3$, the number of perfect matchings in r-regular bipartite graphs grows exponentially, which proves a conjecture by Erdos-Renyi [6]. Schrijver [29] improved EFW inequality to r-regular bipartite graphs, whose asymptotic growth is best possible. Recently, the second name author [19] improved Schrijver's inequality. Moreover, the proof in [19] is significantly simpler and transparent. One of the main tools in the proof in [19] is the use of the classical theory of hyperbolic polynomials.

It was shown by the first name author that $\mathrm{perm}\,_m A \ge \mathrm{perm}\,_m J_n$ for any $A \in \Omega_n$, and for $m \in [2,n]$ equality holds only if and only if $A = J_n$ [9]. ($\mathrm{perm}\,_1 A = n$ for each $A \in \Omega_n$.) This was fact was conjectured by Tverberg [31], and is called in this paper the Friedland-Tverberg (FT) inequality. FT inequality gives a lower bound on the number of partial matchings in any $G \in \Gamma(n,r)$.

We derive here the Schrijver type inequalities for m matchings in r-regular bipartite graphs on $2n$ vertices. This is done using the results and techniques of [19]. In particular we give a generalized versions of FT inequality to positive homogeneous hyperbolic polynomials, which are of independent interest.

The notion of partial matching in $\Gamma(n,r)$ can be extended to asymptotic matchings as $n\to\infty$ as follows. Given a sequence of $G_l \in \Gamma(n_l,r)$ we can consider the quantities

$$f(p,\{G_l\}) := \liminf_{n_l\to\infty} \frac{\log \mathrm{perm}_{m_l} A(G_l)}{2n_l}, \quad F(p,\{G_l\}) := \limsup_{n_l\to\infty} \frac{\log \mathrm{perm}_{m_l} A(G_l)}{2n_l}, \quad (1.1)$$
$$\text{where } n_l \to \infty \text{ and } \lim_{l\to\infty} \frac{m_l}{n_l} = p \in [0,1].$$

$f(p,\{G_l\})$ and $F(p,\{G_l\})$ can be viewed as the minimal and the maximal exponential growth of matchings of density p of the sequence $G_l, l \in \mathbb{N}$.

Consider the following special case of the above example. Let C_m be a cycle on m vertices. Note that C_m is bipartite if and only if m s even. Fix a positive integer d and let $T_{2l,d} := \underbrace{C_{2l}\times\ldots\times C_{2l}}_{d}$, be bipartite toroidal grid on $(2l)^d$ vertices. Note that $T_{2l,d} \in \Gamma(2^{d-1}l^d, 2d)$. It is shown in [21] that $f(p,\{T_{2l,d}\}) = F(p,\{T_{2l,d}\})$ and this quantity is the exponential growth rate of the number of monomer-dimer tilings of the d-dimensional cubic grid having sides of length n, as n tends to infinity and the dimer density, (fraction of the maximum possible number $\frac{1}{2}n^d$ of dimers), in these tilings converges to a fixed number $p \in [0,1]$. (See also [14].) We denote by this exponential growth by $h_d(p)$, and call it the d-dimensional monomer-dimer entropy of dimer density $p \in [0,1]$ in the lattice $\mathbb{Z}^d$, see [21] and [13]. For $d = 1$ the rate is known explicitly as a function p, whereas for $d > 1$ the exact rate is unknown.

$h_d(p)$ can be estimated if one can estimate the quantities $f(p,\{G_l\}) \le F(p,\{G_l\})$ from below and above for any sequence $G_l \in \Gamma(n_l,r), l \in \mathbb{N}$. Lower and upper estimates of $f(p,\{G_l\}) \le F(p,\{G_l\})$ are conjectured in [10] and called the *asymptotic lower matching conjecture* and *asymptotic upper matching conjecture*, abbreviated here by ALMC and AUMC respectively. For $r = 2$ ALMC and AUMC are proved in [11].

In this paper we apply our lower bounds on the sum of subpermanents of doubly stochastic matrices with r nonzero elements in each row to obtain lower bound on $f(p, \{G_l\})$. For a fixed integer $r \geq 3$, we show the validity of ALMC for the densities $p_s = \frac{r}{r+s}$ where $s = 0, 1, \ldots$. These inequalities yield new lower bounds for the d-dimensional monomer-dimer entropy of dimer density $h_d(p), p \in [0, 1]$ in the lattice $\mathbb{Z}^d$. In particular we obtain the best known lower bound for the three dimensional monomer dimer entropy h_3, which combined with the known upper bound in [13] gives the tight result $h_3 \in [.7845, .7863]$.

Next we discuss briefly the sum of $m \times m$ subhafnians of $2n \times 2n$ symmetric B with nonnegative entries, denoted by $\mathrm{haf}\,_m B$. For $0-1$ matrix B this is equivalent to the number of m-matchings in a general graph on $2n$ vertices. We give Ryser-type algorithm for $\mathrm{perm}_m\, A$ and $\mathrm{haf}\,_m B$. Unfortunately, the generating function $\mathbf{x}^\top B\mathbf{x}$, (the quadratic for associated with B), is positive hyperbolic if and only if the second eigenvalue of B is nonpositive. We show that for any graph G, $\mathbf{x}^\top B(G)\mathbf{x}$ is positive hyperbolic if and only if G is a complete k-partite graph. The last section is devoted to algorithmic applications related to $\mathrm{perm}_m\, A$ and $\mathrm{haf}\,_m B$.

We now list briefly the contents of this paper. In §2 we discuss briefly the notion of positive hyperbolic polynomials and their properties that needed here. In §3 we bring the generalized version of FT inequality for positive hyperbolic polynomials. In §4 we state and discuss the ALMC and AUMC. In §5 we give lower bounds on $f(p, \{G_l\})$. We apply these bounds to verify the ALMC for a countable values of densities for each $r \geq 3$ as explained above. In §6 we discuss the notion of $\mathrm{haf}\,_m B$ and its connection to the quadratic form $\mathbf{x}^\top B\mathbf{x}$. In §7 we discuss the Ryser-type formulas for $\mathrm{perm}\,_m A$ and $\mathrm{haf}\,_m B$.

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2 Positive hyperbolic polynomials

Definitions and Notations

1. A vector $\mathbf{x} := (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ is called *positive* or *nonnegative*, and denoted by $\mathbf{x} > \mathbf{0} := (0, \ldots, 0)^\top$ or $\mathbf{x} \geq \mathbf{0}$ if $x_i > 0$ or $x_i \geq 0$ for $i = 1, \ldots, n$ respectively. A nonnegative vector $\mathbf{x} \neq \mathbf{0}$ is denoted by $\mathbf{x} \gneq \mathbf{0}$. $\mathbf{y} \geq \mathbf{x} \iff \mathbf{y} - \mathbf{x} \geq \mathbf{0}$. The cone of all nonnegative vectors in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

2. A polynomial $p = p(\mathbf{x}) = p(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is called *positive hyperbolic* if the following conditions hold:

   - p is a homogeneous polynomial of degree $m \geq 0$.
   - $p(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$.
   - $\phi(t) := p(\mathbf{x} + t\mathbf{u})$, for $t \in \mathbb{R}$, has m-real t-roots for each $\mathbf{u} > \mathbf{0}$ and each $\mathbf{x}$.

3. For any polynomial $p : \mathbb{R}^n \to \mathbb{R}$ and any $\mathbf{0} \neq \mathbf{u} = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$ let $p_\mathbf{u} = p_\mathbf{u}(\mathbf{x}) : \sum_{i=1}^n u_i \frac{\partial p}{\partial x_i}(\mathbf{x})$.

4. Let $\mathbf{e}_i := (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n,\ i = 1, \ldots, n$ be the standard basis in $\mathbb{R}^n$.

5. Let $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$ and denote by $J_n \in \mathbb{R}^{n \times n}$ the $n \times n$ matrix whose all entries are equal to $\frac{1}{n}$.

The following lemma summarizes the basic properties of positive hyperbolic polynomials that needed here.

Lemma 2.1 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \geq 1$. Then the following properties hold:*

1. *Let $\mathbf{u} \gneqq \mathbf{0}, \mathbf{x}$ be fixed and denote $\phi(t) = p(\mathbf{x} + t\mathbf{u})$. Assume that $p(\mathbf{u}) > 0$. Then $\phi(t)$ has m real t roots. Furthermore $p_{\mathbf{u}}(\mathbf{x})$ is a positive hyperbolic polynomial of degree $m-1$. $\mathbf{y} \geq \mathbf{x} \geq \mathbf{0} \Rightarrow p(\mathbf{y}) \geq p(\mathbf{x}) \geq 0$.*

2. *Let $\mathbf{u} \gneqq \mathbf{0}, \mathbf{x} \gneqq \mathbf{0}$ and assume that $p(\mathbf{u}) = 0$. Then either $\phi(t) > 0$ for all $t \geq 0$ or $p(\mathbf{x}) = 0$ and $\phi(t) \equiv 0$. Assume that $p(\mathbf{x}) > 0$ and $\phi(t)$ is not a constant polynomial. Then all its roots are real and negative. If $p_{\mathbf{u}}$ is not a zero polynomial then $p_{\mathbf{u}}$ is a positive hyperbolic of degree $m-1$.*

3. *If $q((x_1, \ldots, x_{n-1})) := p((x_1, x_2, \ldots, x_{n-1}, 0))$ is not identically zero then q is a positive hyperbolic of degree m in $\mathbb{R}^{n-1}$. In particular, $r((x_1, \ldots, x_{n-1})) :=$ $\frac{\partial p}{\partial x_n}((x_1, \ldots, x_{n-1}, 0))$ is either a zero polynomial or a positive hyperbolic polynomial in $n-1$ variables of degree $m-1$.*

Proof.

We assume that $n > 1$ otherwise all the results are trivial

1. The facts $\phi(t)$ has m negative roots and $p_{\mathbf{u}}$ is positive hyperbolic is in [17] or [24]. Hence $p_{\mathbf{u}} \geq 0$ on $\mathbb{R}^n_+$. Let $\mathbf{v} \geq \mathbf{0}$ and assume that $\mathbf{u}_l > \mathbf{0}, l = 1, \ldots$ be a sequence of vectors converging to $\mathbf{v}$. Then $p_{\mathbf{u}_l} \to p_{\mathbf{v}}$. Therefore $p_{\mathbf{v}} \geq 0$ on $\mathbb{R}^n_+$. In particular $p_{\mathbf{e}_i}(\mathbf{x}) = \frac{\partial p}{\partial x_i}(\mathbf{x}) \geq 0$ n $\mathbb{R}^n_+$ for $i = 1, \ldots, n$. Thus $p(\mathbf{x})$ is a nondecreasing function in each variable x_i on R^n_+. Hence $\mathbf{y} \geq \mathbf{x} \geq 0 \Rightarrow p(\mathbf{y}) \geq p(\mathbf{x}) \geq 0$.

2. Recall Brunn-Minkowski theorem that $f := p^{\frac{1}{m}}$ is convex on all positive vectors in $\mathbb{R}^n$ [24, Thm 2, 4)]. Since f is continuous on $\mathbb{R}^n_+$, it follows that f is nonnegative and convex on $\mathbb{R}^n_+$. If $p(\mathbf{x}) = 0$ it follows that $f(\mathbf{y}) = 0$ on the interval joining $\mathbf{x}, \mathbf{u}$. Hence $\phi(t) \equiv 0 \iff p(\mathbf{x}) = 0$. Assume that $p(\mathbf{x}) > 0$. Then $\phi(0) > 0$. If $\phi(t)$ is constant then $\phi(t) = \phi(0) > 0$. Assume that $\phi(t)$ is a nonconstant polynomial. Let $\mathbf{u}_l > \mathbf{0}, l = 1, \ldots$ be a sequence of vectors converging to $\mathbf{u}$. Then $\phi_l(t) := p(\mathbf{x} + t\mathbf{u}_l) \to \phi(t)$. Each ϕ_l has m real negative zeros. Consider the complex projective space $\mathbb{CP}^{n-1}$ with the homogenous coordinates $\mathbf{z} = (z_1, \ldots, z_n)^\top$. The m-roots of $\phi_l(t)$ correspond to the intersection points of the line $L_l := \{\mathbf{z} := s\mathbf{x} + t\mathbf{u}_l,\ s, t \in \mathbb{C}\}$ with the hypersurface $H = \{z \in \mathbb{CP}^n :\ p(\mathbf{z}) = 0\}$ in $\mathbb{CP}^{n-1}$ of degree m. All these points are real and are located in the affine part of of the line L_l, i.e. $s = 1$ and $t < 0$. As $L_l \to L := \{\mathbf{z} := s\mathbf{x} + t\mathbf{u},\ s, t \in \mathbb{C}\}$ it follows that $L_l \cap H \to L \cap H$, counting with the multiplicities if $\phi(t)$ is not constant. Thus all roots of $\phi(t)$ are nonnegative. Since $\phi(0) > 0$ it follows that all the roots of $\phi(t)$ are negative. Hence $\phi(t) > 0$ for $t \geq 0$.

   Assume that $p_{\mathbf{u}} \not\equiv 0$. Recall that $p_{\mathbf{u}_l} \to p_{\mathbf{u}}$. According to part *1* $p_{\mathbf{u}_l}(\mathbf{y}) \geq \mathbf{p}_{\mathbf{u}_l}(\mathbf{x}) \geq 0$ for each pair $\mathbf{y} \geq \mathbf{x} \geq \mathbf{0}$. Hence $p_{\mathbf{u}}(\mathbf{y}) \geq p_{\mathbf{u}}(\mathbf{x}) \geq 0$. If $p_{\mathbf{u}}(\mathbf{y}) = 0$ for some $\mathbf{y} > \mathbf{0}$ then for any $\mathbf{x} \geq 0$, there exists $t > 0$ such that $\mathbf{y} \geq t\mathbf{x} \geq 0$. Hence $t^{m-1}\mathbf{p}_{\mathbf{u}}(\mathbf{x}) = p_{\mathbf{u}}(t\mathbf{x}) = 0 \Rightarrow p_{\mathbf{u}}(\mathbf{x}) = 0 \Rightarrow p_{\mathbf{u}} \equiv 0$ contrary to our assumption. Thus $p(\mathbf{y}) > 0$. As $p_{\mathbf{u}_l}(\mathbf{w} + t\mathbf{y}) \to \psi(t) := p_{\mathbf{u}}(\mathbf{w} + ty)$ and $\psi(t)$ is a polynomial of degree $m-1$, it follows that $\psi(t)$ has $m-1$ real roots for each $\mathbf{w} \in \mathbb{R}^n$. Thus $p_{\mathbf{u}}$ is positive hyperbolic.

3. Since $p(\mathbf{y}) \geq p(\mathbf{x}) \geq 0$ for $\mathbf{y} \geq \mathbf{x}$ on $\mathbb{R}^n_+$ it follows that $q(\mathbf{y}_1) \geq q(\mathbf{x}_1) \geq 0$ on $\mathbb{R}^{n-1}_+$, where $\mathbf{y} = (\mathbf{y}_1^\top, 0)^\top, \mathbf{x} = (\mathbf{x}_1^\top, 0)^\top$. Since q is a homogeneous of degree m, from the arguments in part *2* it follows that either $q(\mathbf{y}_1) > 0$ for each $\mathbf{y}_1 > \mathbf{0}$ or $q \equiv 0$. Assume that $p(\mathbf{y}) = q(\mathbf{y}_1) > 0$. Then according to part *1*, $p(\mathbf{x} + t\mathbf{y}) = q(\mathbf{x}_1 + t\mathbf{y}_1)$ has m real roots in t. Assume that $r \not\equiv 0$. Hence $p_{\mathbf{e}_n} \not\equiv 0$. So $p_{\mathbf{e}_n}$ is positive hyperbolic by *2*, and by previous arguments r is positive hyperbolic on $\mathbb{R}^{n-1}$.

□

The following propositions are well known and we bring their proof for completeness.

Proposition 2.2 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \geq 1$. Then the coefficient of each monomial in p is nonnegative.*

Proof. Let p be a positive hyperbolic polynomial of degree $m \geq 1$. Let $\mathbf{u} > \mathbf{0}$. Part *1* of the above lemma yields that $p_{\mathbf{u}}$ is positive hyperbolic of degree $m-1$.

We prove the proposition by induction on m. Let $m = 1$. Assume that $\mathbf{u} > \mathbf{0}$. Then $p_{\mathbf{u}} = \nabla p \mathbf{u} > 0$. Hence $\nabla p \geq \mathbf{0}$ and the corollary holds.

Assume that the result hold for $m = l \geq 1$. Let p be a positive hyperbolic polynomial of degree $m = l+1$. Let $\mathbf{e}_i = (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n$ for $i = 1, \ldots, n$. Let $\mathbf{u}_j > \mathbf{0}, j = 1, \ldots,$ and assume that $\lim_{j\to\infty} \mathbf{u}_j = \mathbf{e}_i$. Hence $p_{\mathbf{u}_j}$ is positive hyperbolic of degree l. By the induction hypothesis the coefficients of all monomials of $p_{\mathbf{u}_j}$ are nonnegative. Let $j \to \infty$ and deduce that the coefficients of all monomials of $p_{\mathbf{e}_i}$ are nonnegative. Hence the coefficients of all monomials of p which include the variable x_i are nonnegative. Let $i = 1, \ldots, n$ to deduce that the coefficients of all monomials of degree one at least are nonnegative. As $p(\mathbf{0}) \geq 0$ we deduce the proposition. $\square$

Proposition 2.3 *For $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ let $e^{\mathbf{x}} := (e^{x_1}, \ldots, e^{x_n})^\top > \mathbf{0}$. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a nonzero polynomial with such that the coefficient of each monomial is nonnegative. Then $\log p(e^{\mathbf{x}})$ is a convex function on $\mathbb{R}^n$. Let $L \subseteq \mathbb{R}^n$ be a line in $\mathbb{R}^n$. Then the restriction of $\log p(e^{\mathbf{x}})$ to L is either an affine function or a strictly convex function.*

Proof. The convexity of $\log p(e^{\mathbf{x}})$ can be found in [23]. Note that $f := p(e^{\mathbf{x}})|L$ is of the form in $\sum_{i=1}^k a_i e^{b_i t}$, where each $a_i > 0$ and $b_1 > b_2 > \ldots > b_k$. If $k = 1$ then $\log f(t)$ is $\log a_1 + b_1 t$. Otherwise it straightforward to show that $\frac{f'}{f}$ is increasing on $\mathbb{R}$. $\square$

Corollary 2.4 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \geq 1$. Then $\log p(e^{\mathbf{x}})$ is a convex function on $\mathbb{R}^n$. Let $L \subseteq \mathbb{R}^n$ be a line in $\mathbb{R}^n$. Then the restriction of $\log p(e^{\mathbf{x}})$ to L is either an affine function or a strictly convex function.*

Examples of positive hyperbolic polynomials

1. Let $A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}^{m\times n}$ be a nonnegative matrix, denoted by $A \geq \mathbf{0}$, where each row of A is nonzero. Fix a positive integer $k \in [1, m]$. Then

$$p_{k,A}(\mathbf{x}) := \sum_{1\leq i_1 < \ldots i_k \leq m} \prod_{j=1}^k (A\mathbf{x})_{i_j}, \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

   is positive hyperbolic of degree k in n variables.

2. Let $A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ hermitian, nonnegative definite matrices such that $A_1 + \ldots + A_n$ is a positive definite matrix. Let $p(\mathbf{x}) = \det \sum_{i=1}^n x_i A_i$. Then $p(\mathbf{x})$ is positive hyperbolic.

Proof.

1. First note that $p_{k,A}(\mathbf{x}) > 0$ for $\mathbf{x} > \mathbf{0}$. The hyperbolicity of $p_{m,A}$ and $p_{1,A}$ is obvious. Assume that $k \in (1, m)$. Let $\mathbf{z} = (z_1, \ldots, z_{n+m-k})^\top \in \mathbb{R}^{n+m-k}$ and define $P(\mathbf{z}) := \prod_{i=1}^m (\sum_{j=1}^n a_{ij} z_j + \sum_{j=n+1}^{n+m-k} z_j)$. Then

$$p_{k,A}(\mathbf{x}) = \binom{m}{k}^{-1} \frac{\partial^{m-k} P}{\partial z_{n+1} \ldots \partial z_{n+m-k}}((x_1, \ldots, x_n, 0, \ldots, 0)).$$

   Hence by Lemma 2.1 $p_{k,A}$ positive hyperbolic.

2. This is a standard example and the proof is straightforward.

□

Let $p(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \ge 1$. As in [19] define

$$\mathrm{Cap}\, p := \inf_{\mathbf{x}>\mathbf{0}, x_1 \dots x_n = 1} p(\mathbf{x}) = \inf_{\mathbf{x}>\mathbf{0}} \frac{p(\mathbf{x})}{(x_1 \dots x_n)^{\frac{m}{n}}}. \tag{2.2}$$

It is possible that $\mathrm{Cap}\, p = 0$. For example let $p = x_1^{m_1} \dots x_n^{m_n}$ where $m_1, \dots, m_n$ are nonnegative integer whose sum is m and $(m_1, \dots, m_n) \ne k\mathbf{1}$.

Proposition 2.5 *Let $A \in \mathbb{R}^{n\times n}$ be a doubly stochastic matrix. Let $p_{k,A}, k \in [1, n]$ be positive hyperbolic defined as part 1 of the above example. Then $\mathrm{Cap}\, p_{k,A} = \binom{n}{k}$. Let $B \in \mathbb{R}^n$ be a matrix with positive entries. Then there exists two positive definite diagonal matrices D_1, D_2, unique up to $tD_1, t^{-1}D_2, t > 0$, such that $A := D_1 B D_2$ is a doubly stochastic matrix [30]. Let $p_{n,B}$ be defined as above. Then $\mathrm{Cap}\, p_{n,B} = \frac{1}{\det D_1 D_2}$.*

Proof. Consider first $p_{n,A}$. Since A is row stochastic $p_{n,A}(\mathbf{1}) = 1$. Hence $\mathrm{Cap}\, p_{n,A} \le 1$. Let $\mathbf{u} = (u_1, \dots, u_n)^\top \gneqq \mathbf{0}$ be a probability vector. Then for any $\mathbf{x} = (x_1, \dots, x_n) > \mathbf{0}$ the generalized arithmetic-geometric inequality states $\mathbf{u}^\top \mathbf{x} \ge \prod_{i=1}^n x_i^{u_i}$. Use this inequality for each $(A\mathbf{x})_i$. The assumption that A is doubly stochastic yields that $p_{n,A} \ge x_1 \dots x_n \Rightarrow \mathrm{Cap}\, p_{n,A} \ge 1$. Hence $\mathrm{Cap}\, p_{n,A} = 1$.

Let $k \in [1, n)$. Then $p_{k,A}(\mathbf{1}) = \binom{n}{k}$. Hence Hence $\mathrm{Cap}\, p_{k,A} \le \binom{n}{k}$. Apply the arithmetic-geometric inequality to $\frac{p_{k,A}}{\binom{n}{k}}$ to deduce that $p_{k,A} \ge \binom{n}{k} p_{n,A}^{\frac{m}{n}}$. Hence $\mathrm{Cap}\, p_{k,A} \ge \binom{n}{k}$.

It is straightforward to show that $\frac{p_{n,B}(\mathbf{x})}{x_1 \dots x_n} = \frac{p_{n,A}(\mathbf{y})}{\det(D_1 D_2) y_1 \dots y_n}$, where $\mathbf{y} = D_2^{-1}\mathbf{x}$. Hence $\mathrm{Cap}\, p_{n,B} = \frac{1}{\det D_1 D_2}$. □

The following result is taken from [19].

Lemma 2.6 *Let $k \ge 1$ be an integer, $\mathbf{u} := (u_1, \dots, u_k)^\top > \mathbf{0}, \mathbf{v} := (v_1, \dots, v_k)^\top > \mathbf{0}$ and define $f(t) := \prod_{i=1}^k (u_i t + v_i)$. Let $K(f) := \inf_{t>0} \frac{f(t)}{t}$. Then $f'(0) = K$ for $k = 1$ and $f'(0) \ge (\frac{k-1}{k})^{k-1} K$ for $k \ge 2$. For $k \ge 2$ equality holds if and only if $\frac{v_1}{u_1} = \dots = \frac{v_k}{u_k}$.*

Proof. We can assume WLOG that $f(0) = 1$; i.e. that $f(t) := \prod_{i=1}^k (a_i t + 1), a_i = \frac{u_i}{v_i}$. Using the arithmetic-geometric means inequality we get that

$$Kt \le f(t) \le p(t) =: (1 + \frac{f'(0)}{k} t)^k, t \ge 0.$$

Therefore , by doing basic calculus ,

$$K \le \inf_{t>0} \frac{p(t)}{t} = f'(0) (\frac{k}{k-1})^{k-1},$$

which finally gives the desired inequality

$$f'(0) \ge (\frac{k-1}{k})^{k-1} K, k \ge 2.$$

It follows again from arithmetic-geometric means inequality that the equality holds if and only if $a_1 = \frac{v_1}{u_1} = \dots = a_k = \frac{v_k}{u_k}$. □

Definition. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree m. For each integer $i \in [0, n]$ the *i-th degree* of p is the integer $r_i \in [0, m]$ such that

$$\frac{\partial^{r_i} p}{\partial x_i^{r_i}}(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \not\equiv 0, \text{ and } \frac{\partial^{r_i+1} p}{\partial x_i^{r_i+1}}(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \equiv 0.$$

Let $\deg_i p := r_i$ for $i = 1, \dots, n$.

The following proposition follows straightforward from part *3* of Lemma 2.1.

Proposition 2.7 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree m. Let $i \in [1, n]$ be an integer. Then*

1. $\deg_i p = 0 \iff p(\mathbf{x}) = (p(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n))$.

2. *For each integer $j \in [0, \deg_i p]$ $\frac{\partial^j p}{\partial x_i^j}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ is a positive hyperbolic polynomial of degree $m - j$.*

3. *For each integer $j \in [1, n], j \neq i$,*

$$\deg_j \frac{\partial p}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \leq \min(\deg_j p, n - 1).$$

The following result is crucial for the proof of a generalized Friedland-Tverberg inequality and is due essentially to the second author in [19].

Lemma 2.8 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \geq 1$. Assume that* $\mathrm{Cap}\, p > 0$. *Then* $\deg_i p \geq 1$ *for $i = 1, \ldots, n$. For $m = n \geq 2$*

$$\mathrm{Cap}\, \frac{\partial p}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geq (\frac{\deg_i p - 1}{\deg_i p})^{\deg_i p - 1} \mathrm{Cap}\, p \text{ for } i = 1, \ldots, n,$$

where $0^0 = 1$.

Proof. It is enough to prove the result for $i = n$. Suppose to the contrary that p does not depend on x_n. Then let $\mathbf{x}(t) = (1, \ldots, 1, t)^\top$ and $t \to \infty$ in (2.2) to deduce that $\mathrm{Cap}\, p = 0$ contrary to our assumption.

Assume that $m = n > 1$. Let $k = \deg_n p \geq 1$. Let $\mathbf{x}_0 := (x_1, \ldots, x_{n-1}, 0)^\top, \mathbf{x}_1 := (x_1, \ldots, x_{n-1})^\top$. Proposition 2.7 yields that $g(\mathbf{x}_1) := \frac{\partial^k p}{\partial x_i^k}(\mathbf{x}_0)$ is a positive hyperbolic function in $n - 1$ variables of degree $m - l$. Hence $g(\mathbf{x}_1) > 0$ for $\mathbf{x}_1 > \mathbf{0}$. Thus for $\mathbf{x}_1 > \mathbf{0}$

$$p(\mathbf{x}_0 + t\mathbf{e}_n) = k! g(\mathbf{x}_1) t^l + \ldots = k! g(\mathbf{x}_1) \prod_{i=1}^{k} (t + \lambda_i(\mathbf{x}_1)), \quad \lambda_i(x) > 0, \text{ for } i = 1, \ldots, k. \tag{2.3}$$

The second equality follows from part *2* of Lemma 2.1. Assume in addition that $x_1 \ldots x_{n-1} = 1$. Then $\inf_{t>0} \frac{p(\mathbf{x}_0 + t\mathbf{e}_n)}{t} \geq \mathrm{Cap}\, p$. Apply Lemma 2.6 to the right-hand side of (2.3) to deduce that $\frac{\partial p}{\partial x_n}(\mathbf{x}_0) \geq (\frac{k-1}{k})^{k-1} \mathrm{Cap}\, p$. Since we assumed that $x_1 \ldots x_{n-1} = 1$ it follows that $\mathrm{Cap}\, \frac{\partial p}{\partial x_n}(\mathbf{x}_0) \geq (\frac{k-1}{k})^{k-1} \mathrm{Cap}\, p$. $\square$

Remark 2.9 *Lemma 2.6 , which is simple but crucial , is a particular case of the following general result :*
Let $f : [0, \infty) \to R_+$ be a nonnegative function differentiable at zero from the right ; $K = \inf_{t>0} \frac{f(t)}{t}$. If $k \geq 1$ and $f^{\frac{1}{k}}$ is concave then $f'(0) \geq (\frac{k-1}{k})^{k-1} K$. On the other hand if $k \geq 1$ and $f^{\frac{1}{k}}$ is convex then $f'(0) \leq (\frac{k-1}{k})^{k-1} K$.

3 Friedland-Tverberg inequality

Theorem 3.1 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1, n]$. Assume that $\deg_i p \leq r_i \in [1, m]$ for $i = 1, \ldots, n$. Rearrange the sequence $r_1, \ldots, r_n$ in an increasing order $1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^*$. Let $k \in [1, n]$ be the smallest integer such that $r_k^* > m - k$. Then*

$$\sum_{1\le i_1<\ldots<i_m\le n}\frac{\partial^m p}{\partial x_{i_1}\ldots\partial x_{i_m}}(\mathbf{0})\ge$$
$$\frac{n^{n-m}}{(n-m)!}\frac{(n-k+1)!}{(n-k+1)^{n-k+1}}\prod_{j=1}^{k-1}\Big(\frac{r_j^*+n-m-1}{r_j^*+n-m}\Big)^{r_j^*+n-m-1}\mathrm{Cap}\, p. \tag{3.1}$$

(Here $0^0=1$, and the empty product for $k=1$ is assumed to be 1.) If Cap >0 *and $r_i=m$ for $i=1,\ldots,m$ equality holds if and only if $p=C(\frac{x_1+\ldots+x_n}{n})^m$ for each $C>0$.*

Proof. Suppose that Cap $p=0$. Then part *3* of Lemma 2.1 yields that the left-hand side of (3.1) is nonnegative and the theorem holds in this case.

Clearly, it is enough to assume the case Cap $p=1$. The case $m=n$ is essentially proven in [19] and we repeat its proof for the convenience of the reader. Permute the coordinates of $x_1,\ldots,x_n$ such that $\deg_n p=\min_{i\in[1,n]}\deg_i p\le r_1^*$. Assume that $\deg_n p=l$. Then Lemma 2.8 yields that $r((x_1,\ldots,x_{n-1})):=\frac{\partial p}{\partial x_n}((x_1,\ldots,x_{n-1},0))$ is positive hyperbolic of degree $n-1$ and Cap $r\ge(\frac{l-1}{l})^{l-1}$Cap p. Since the sequence $(\frac{i-1}{i})^{i-1}, i=1,\ldots,$ is decreasing to have the lowest possible lower bound we have to assume $l=r_1^*$. Suppose first that $r_1^*=n$. Repeating this process n times we get that

$$\frac{\partial^n p}{\partial x_1\ldots\partial x_n}(\mathbf{0})\ge \mathrm{Cap}\, p\prod_{j=2}^{n}\Big(\frac{j-1}{j}\Big)^{j-1}=\frac{n!}{n^n}\mathrm{Cap}\, p.$$

This inequality to corresponds to the case $r_i^*=n$ for $i=1,\ldots,n$. The equality case is discussed in [19].

Let $m\in[1,n-1]$. Put $P(\mathbf{x})=p(\mathbf{x})(\frac{1}{n}\sum_{i=1}^n x_i)^{n-m}$. Clearly, P is positive hyperbolic of degree n. Since $\frac{1}{n}\sum_{i=1}^n x_i\ge(x_1\ldots x_n)^{\frac{1}{n}}$ for each $\mathbf{x}\ge 0$, it follows that Cap $P\ge$ Cap p. Apply (3.1) to P for $m=n$ to deduce (3.1) in the general case. Since the equality case for P holds if and only $P=(\frac{1}{n}\sum_{i=1}^n x_i)^n$ it follows that the equality in (3.1) holds if and only if $p=(\frac{1}{n}\sum_{i=1}^n x_i)^m$. □

Let $A\in\mathbb{R}^{n\times n}$ be a doubly stochastic matrix. Apply this theorem to $p_{m,A}$ defined Proposition 2.5 to deduce the Friedland-Tverberg inequality for the sum of all $m\times m$ permanents of A:

Corollary 3.2 *Let $A\in\mathbb{R}_+^{n\times n}$ be a doubly stochastic matrix. Then* perm $_mA\ge\binom{n}{m}^2\frac{m!}{n^m}$ *for any $m\in[2,n]$. equality holds if and only $A=J_n$.*

Theorem 3.3 *(Gurvits) Let $A\in\mathbb{R}_+^{n\times n}$ be a doubly stochastic matrix, such that each column contains at most $r\in[1,n]$ nonzero entries. Then*

$$\mathrm{perm}\, A\ge\frac{r!}{r^r}\Big(\frac{r-1}{r}\Big)^{(r-1)(n-r)}=\frac{r!}{r^r}\Big(\frac{r}{r-1}\Big)^{r(r-1)}\Big(\frac{r-1}{r}\Big)^{(r-1)n}. \tag{3.2}$$

Proof. Note that for $p(\mathbf{x})=\prod_{i=1}^n(A\mathbf{x})_i$ we have that $\deg_i p=r$ for $i=1,\ldots,n$. Apply (3.1) to this case, i.e. $m=n$, $r_j^*=r, j=1,\ldots,n$ and $k=n-r+1$ to deduce the theorem. □

4 The ALMC and AUMC

Let $G=(V,E)$ be a general graph with the set of vertices V and edges E. A *matching* in G is a subset $M\subseteq E$ such that no two edges in M share a common endpoint. The endpoints of the edges in M are said to be *covered* by M. We can think of each edge $e=(u,v)\in M$ as

occupied by a *dimer*, consisting of two neighboring atoms at u and v forming a bond, and of each vertex not covered by M as a *monomer*, which is an atom not forming any bond. For this reason a matching in G is also called a *monomer-dimer cover* of G. If there are no monomers, M is said to be a *perfect matching*. Note that if a perfect matching exists then $\#V$ is even. A matching M with $\#M = k$ is called an *k-matching*. We denote by $\phi_G(k)$ be the number of k-matchings in G (in particular $\phi_G(0) = 1$), and by $\Phi_G(x) := \sum_k \phi_G(k)x^k$ the matching generating polynomial of G. It is known that all the roots of matching polynomial are real negative numbers [27].

Let G be a bipartite graph $G = (V, E)$, where $V = V_1 \cup V_2$ is the set of vertices of G and E is the set of edges that connect vertices in V_1 to vertices in V_2. Assume that $\#V_1 = \#V_2 = n$. We identify V_1 and V_2 with $\langle n \rangle := \{1, \ldots, n\}$, where the vertices in V_1 and V_2 are colored with colors black and white respectively. Then G is represented by $0-1$ $n \times n$ matrix $A(G) = A = (a_{ij}) \in \{0, 1\}^{n\times n}$, where $a_{ij} = 1$ if and only if the black vertex i is connected to the white vertex j. It is convenient to consider multi bipartite graphs. Thus, the entries of the representation matrix $A(G) = (a_{ij}) \in \mathbb{Z}_+^{n\times n}$ are nonnegative integers, where a_{ij} is the number of edges from the black vertex i to the white vertex j.

It is straightforward to show that

$$\phi_G(k) = \mathrm{perm}\,_k A(G), \quad k = 0, \ldots, n, \text{ where } \mathrm{perm}\,_0 A := 1 \text{ for any } A \in \mathbb{R}^{n\times n}. \tag{4.1}$$

Let $\Gamma(n, r)$ be the set of bipartite r-regular multi-graphs, (each vertex has degree r), with $n := \frac{\#V}{2}$. Let $\Delta(n, r)$ be the set by an $n \times n$ nonnegative matrices A with integer entries, such that the sum of each row and column is r. Then each $G \in \Gamma(n, r)$ is represented by $A \in \Delta(n, r)$ and vice versa. Note for each $A \in \Delta(n, r)$ the matrix $\frac{1}{r}A$ is doubly stochastic. Corollary 3.2 yields:

$$\phi_G(m) \geq \binom{n}{m}^2 \frac{m! r^m}{n^m} \quad \text{for any } G \in \Gamma(n, r). \tag{4.2}$$

Note that the symmetric group S_n on n elements, presented as the group of permutation matrices $\Pi_n \subset \{0, 1\}^{n\times n}$ acts from the left and from the right on $\Delta(n, r)$, i.e. $P\Delta(n, r) = \Delta(n, r)P$ for each $P \in \Pi_n$. These actions are equivalent to the action of S_n on V_1 and V_2 respectively.

There is a standard probabilistic model on $\Gamma(n, r)$, which assigns a fairly natural probability measure $\nu(n, r)$ on $\Gamma(n, r)$ [27]. The measure $\nu(n, r)$ is invariant under the action of S_n on V_1 and V_2 as explained above. By abuse of the notation we view $\nu(n, r)$ also a probability measure on $\Delta(n, r)$, which is invariant under the left and the right action of Π_n. The following result is proven in [11]:

Theorem 4.1 *Let $\nu(n, r)$ be the probability measure defined above. Then*

$$E_{\nu(n,r)}(\phi(G, m)) = E_{\nu(n,r)}(\mathrm{perm}\,_m A) = \frac{\binom{n}{m}^2 r^{2m} m!(rn - m)!}{(rn)!}, \quad m = 0, \ldots, n. \tag{4.3}$$

In particular, let $k_n \in [0, n]$, $n = 1, 2, \ldots$ be a sequence of integers with $lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$. Then

$$\lim_{n\to\infty} \frac{\log E_{\nu(n,r)}(\phi(G, k_n))}{2n} = gh_r(p), \tag{4.4}$$

where

$$gh_r(p) := \tfrac{1}{2}\left(p \log r - p \log p - 2(1 - p) \log(1 - p) + (r - p) \log\left(1 - \tfrac{p}{r}\right)\right), \tag{4.5}$$

The case $m = n$ in (4.3) is given in [27].

Fix as subset $J \subset \langle n \rangle$ of cardinality m: $\#J = m$. For $G \in \Gamma(n,r)$ let $\phi(G,J)$ be all m-matching in G that cover the set $J \subset V_2$. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $I \subset \langle n \rangle$ let $A[J|I]$ be the submatrix $(a_{ij})_{i\in I, j\in J}$. Denote

$$\text{perm}_{\,m} A[\langle n \rangle | J] = \sum_{I\subset\langle n\rangle, \#I=m} \text{perm}_{\,m} A[I|J].$$

Then $\phi(G,J) = \text{perm}_{\,m} A(G)[\langle n \rangle | J]$.

Use the invariance of $\nu(n,r)$ under the action of S_n on V_2 and the fact that there are $\binom{n}{r}$ distinct subsets $J \subset \langle n \rangle$ of cardinality m to obtain:

Corollary 4.2 *Let $\nu(n,r)$ be the probability measure defined above. Then for any $J \subset \langle n \rangle, \#J = m$*

$$E_{\nu(n,r)}(\phi(G,J)) = E_{\nu(n,r)}(\text{perm}_{\,m} A[\langle n \rangle|J]) = \frac{\binom{n}{m} r^{2m} m!(rn-m)!}{(rn)!}, \quad m = 0, \ldots, n. \tag{4.6}$$

The following conjecture is stated in [10].

Conjecture 4.3 (The Asymptotic Lower Matching Conjecture) *For $r \geq 2$, let $G_n = (V_n, E_n)$, $n = 1, 2, \ldots$ be a sequence of finite r-regular bipartite graphs with $\#V_n \to \infty$. Let $k_n \in [0, \frac{\#V_n}{2}]$, $n = 1, 2, \ldots$ be a sequence of integers with $lim_{n\to\infty} \frac{2k_n}{\#V_n} = p \in (0,1]$. Then*

$$\liminf_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n} \geq gh_r(p). \tag{4.7}$$

For $r = 1$ this conjecture holds trivially. For $r = 2$ this conjecture is proved in [11]. The inequality (4.2) implies that under the conditions of Conjecture 4.3 the following inequality holds, see [13]

$$\liminf_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n} \geq fh_r(p), \tag{4.8}$$

where

$$fh_r(p) := \tfrac{1}{2}(-p\log p - 2(1-p)\log(1-p) + p\log r - p). \tag{4.9}$$

As usual, we denote by $\mathbb{R}[x]$ the algebra of polynomials in x with real coefficients, by $0 \in \mathbb{R}[x]$ the zero polynomial, and by $\mathbb{R}_+[x] \subset \mathbb{R}[x]$ the subalgebra of polynomials with non-negative coefficients. We partially order $\mathbb{R}[x]$ by writing, for $f, g \in \mathbb{R}[x]$, $g \succeq f$ when $g - f \in \mathbb{R}_+[x]$, and $g \succ f$ when $g - f \in \mathbb{R}_+[x] \setminus \{0\}$. Clearly, if $g_1 \succeq f_1 \succ 0$ and $g_2 \succeq f_2 \succ 0$, then $g_1 g_2 \succ f_1 f_2$ unless $g_1 = f_1$ and $g_2 = f_2$.

Let $qK_{r,r}$ denote the union of q complete bipartite graphs $K_{r,r}$ having r vertices of each color class. It is straightforward to show that any finite graphs G, G' satisfy

$$\Phi_{G\cup G'}(x) = \Phi_G(x)\Phi_{G'}(x), \tag{4.10}$$

and that

$$\Phi_{K_{r,r}}(x) = \sum_{k=0}^{r} \binom{r}{k}^2 k!\, x^k. \tag{4.11}$$

The following conjecture is stated in [10]

Conjecture 4.4 (The Upper Matching Conjecture) *Let G be a bipartite r-regular graph on $2qr$ vertices where $q, r \geq 2$. Then $\Phi_G \preceq \Phi_{qK_{r,r}}$, equality holding only if $G = qK_{r,r}$.*

For $k = 2$ this conjecture is proved in [11]. The above conjecture implies the following Asymptotic Upper Matching Conjecture [10]. Denote by $K(r)$ be the countably infinite

union of $K_{r,r}$. Let $P_{K(r)}(t)$ and $h_{K(r)}(p), p \in [0,1]$ be the pressure and the p-matching entropy associated and the with $K(r)$ [14]:

$$P_{K(r)}(t) = \frac{\log \sum_{k=0}^{r} \binom{r}{k}^2 k!\, e^{2kt}}{2r}, \qquad t \in \mathbb{R}. \tag{4.12}$$

$$h_{K(r)}(p(t)) = P_{K(r)}(t) - tp(t), \qquad t \in \mathbb{R} \tag{4.13}$$

where

$$p(t) = P'_{K(r)}(t) = \frac{\sum_{k=0}^{r} \binom{r}{k}^2 k!\, (2k) e^{2kt}}{2r \sum_{k=0}^{r} \binom{r}{k}^2 k!\, e^{2kt}}, \qquad t \in \mathbb{R}. \tag{4.14}$$

Conjecture 4.5 (The Asymptotic Upper Matching Conjecture) *For $r \geq 2$, let $G_n = (V_n, E_n)$, $n = 1, 2, \ldots$ be a sequence of finite r-regular bipartite graphs with $\#V_n \to \infty$. Let $k_n \in [0, \frac{\#V_n}{2}]$, $n = 1, 2, \ldots$ be a sequence of integers with $lim_{n\to\infty} \frac{2k_n}{\#V_n} = p \in (0, 1]$. Then*

$$\limsup_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n} \leq h_{K(r)}(p). \tag{4.15}$$

Equality case holds for the sequence $qK_{r,r}, q = 1, 2, \ldots$.

For $r = 2$ the AUMC is proven in [11]. For $p = 1$ and any $r \in \mathbb{N}$ the AUMC follows from the proof of Minc conjecture by Bregman [4]. Some computations performed in [10] support the ALMC and AUMC.

The following plots illustrating the Asymptotic Matching Conjectures for $r = 4, 6$. Let C_n a cycle on n points, and let $T_{n,d} = (V_n, E_n) := \underbrace{C_n \times \ldots \times C_n}_{d}, n = 3, \ldots$ be a sequence of d dimensional torii. Note that each $T_{n,d}$ is $2d$ regular graph. It is a classical result that the following limit exists for any $p \in [0, 1]$:

$$\lim_{n\to\infty} \frac{\log \phi_{T_{n,d}}(k_n)}{\#V_n} = h_d(p), \quad p \in [0, 1]. \tag{4.16}$$

$h_d(p)$ is the d-dimensional monomer-dimer entropy of dimer density $p \in [0, 1]$ in the lattice $\mathbb{Z}^d$ [21] and [13]. In this case we use the notation $h_d := \max_{p\in[0,1]} h_d(p)$, (the quantities h_d and $\tilde{h}_d := h_d(1)$ are called the d-monomer-dimer entropy and the 2-dimer entropy, respectively, in [13]). Figure 1 shows various bounds and values for the monomer-dimer entropy $h_2(p)$ of dimer density $p \in [0, 1]$ in the 4-regular 2-dimensional grid. FT is the Friedland-Tverberg lower bound $fh_4(p)$ of (4.9), h2 is the true monomer-dimer entropy equal to $\max_{p\in[0,1]} h_2(p)$ (it is known to a precision much greater than the picture resolution). The crosses marked B are Baxter's computed values [1]. ALMC is the function $gh_4(p)$ of (4.5), conjectured to be a lower bound in the Asymptotic Lower Matching Conjecture. AUMC is the monomer-dimer entropy $h_K(p)$ of dimer density p in a countable union of $K_{4,4}$, given by (4.12)–(4.14) and conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture. Notice that AUMC goes a little over h2: a countable union of $K_{4,4}$ has a higher monomer-dimer entropy than an infinite planar grid.

Figure 2 shows similarly various bounds and values for the monomer-dimer entropy $h_3(p)$ of dimer density $p \in [0, 1]$ in the 6-regular 3-dimensional grid. FT is the Friedland-Tverberg lower bound $fh_6(p)$ of (4.9), h3Low and h3High are the best known lower and upper bounds for the true monomer-dimer entropy equal to $\max_{p\in[0,1]} h_3(p)$. ALMC is the function $gh_6(p)$ of (4.5), conjectured to be a lower bound in the Asymptotic Lower Matching Conjecture. AUMC is the monomer-dimer entropy $h_K(p)$ of dimer density p in a countable union of $K_{6,6}$, given by (4.12)–(4.14) and conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture. Notice that AUMC goes a little over h3High: a countable union of $K_{6,6}$ has a higher monomer-dimer entropy than an infinite cubic grid.

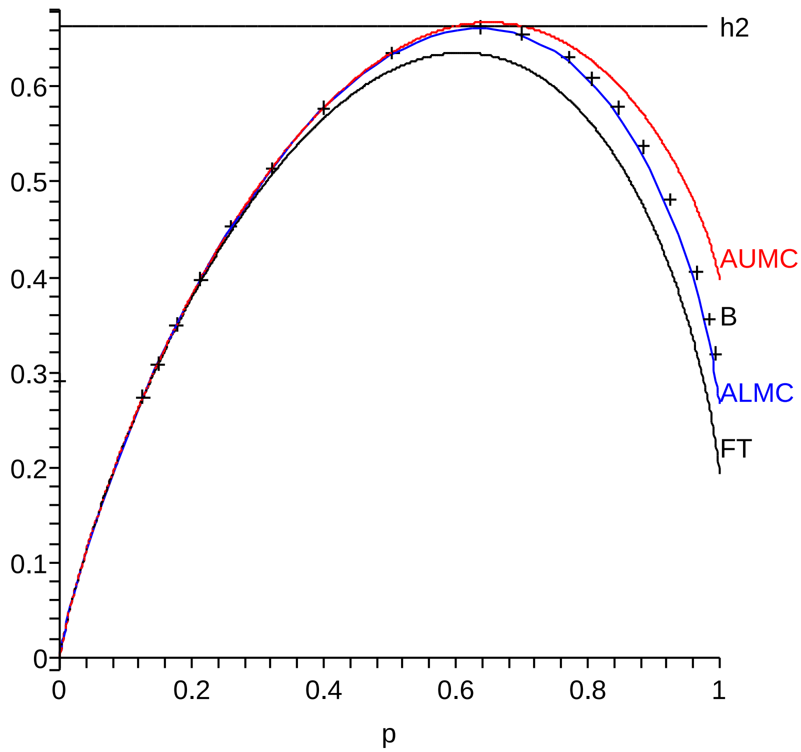

Figure 1: Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

5 A proof of some case of the ALMC

In this section we prove the following case of ALMC:

Theorem 5.1 *Let $r \geq 3$ be an integer. Then the asymptotic lower matching conjecture (4.8) holds for $p_s = \frac{r}{r+s}, s = 0, 1, 2, \ldots$.*

The proof of this theorem follows from the following results.

Theorem 5.2 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1, n)$. Assume that $\deg_i p \leq r_i \in [1, m]$ for $i = 1, \ldots, n$. Rearrange the sequence $r_1, \ldots, r_n$ in an increasing order $1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^*$. Let $s \in \mathbb{N}$. Let $k \in [1, n]$ be the smallest integer such that $r_k^* + s > n - k$. Then*

$$\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) \geq$$

$$\frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left(\frac{r_j^* + s - 1}{r_j^* + s}\right)^{r_j^* + s - 1} \mathrm{Cap}\, p. \qquad (5.1)$$

Proof. Let $q : \mathbb{R}^n \to \mathbb{R}$ be positive hyperbolic of degree $n - m$ with $\deg_i q \leq s$ for $i = 1, \ldots, n$ and $\mathrm{Cap}\, q = 1$. Then $f = pq : \mathbb{R}^n \to \mathbb{R}$ is positive hyperbolic of degree n with

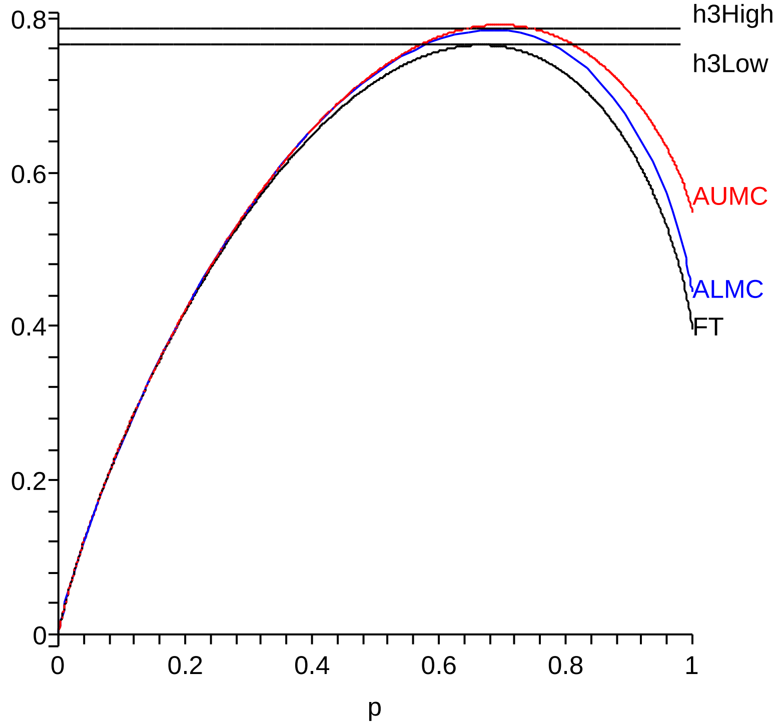

Figure 2: Monomer-dimer tiling of the 3-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h3Low and h3High are the known bounds for the monomer-dimer entropy. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{6,6}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

Cap $f \geq$ Cap p and $\deg_i f \leq r_i + s$ for $i = 1, \ldots, n$. Apply Theorem 3.1 to f to deduce

$$\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) \frac{\partial^{n-m} q}{\partial x_{i'_1} \ldots \partial x_{i'_{n-m}}}(\mathbf{0}) \geq$$

$$\frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left(\frac{r_j^* + s - 1}{r_j^* + s}\right)^{r_j^* + s - 1} \text{Cap}\, p, \tag{5.2}$$

where $1 \leq i'_1 < \ldots < i'_{n-m} \leq n$ and $\{i_1, \ldots, i_m, i'_1, \ldots, i'_{n-m}\} = \langle n \rangle$.

Let $A \in \Delta(n, s)$ and choose $q = \binom{n}{n-m}^{-1} p_{n-m, \frac{1}{s}A}(\mathbf{x})$ as in (2.1). Note

$$\frac{\partial^{n-m} q}{\partial x_{i'_1} \ldots \partial x_{i'_{n-m}}}(\mathbf{0}) = \frac{1}{\binom{n}{n-m} s^{n-m}} \text{perm}_{\,n-m} A[\langle n \rangle | J'].$$

Now take the expected value of the left-hand side of the inequalities (5.2) corresponding to all $A \in \Delta(n, s)$. Use Corollary 4.2 to deduce that the coefficient of each $\frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0})$ is $\frac{s^{n-m}(n-m)!((s-1)n+m)!}{(sn)!}$. □

Corollary 5.3 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1, n)$. Assume that $\deg_i p \le r \in [1, m]$ for $i = 1, \ldots, n$. Let $s \in \mathbb{N}$ and $k = n - r - s + 1 \ge 1$. Then*

$$\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) \ge$$

$$\frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!} \frac{(r+s)!}{(r+s)^{r+s}} (\frac{r+s-1}{r+s})^{(r+s-1)(n-r-s)} \mathrm{Cap}\, p. \tag{5.3}$$

Theorem 5.4 *Let $B \in \mathbb{R}_+^{n \times n}$ be a doubly stochastic matrix with at most r nonzero entries in each column. Let $s \in \mathbb{N}$ and $k = n - r - s + 1 \ge 1$. Then for each $m \in [1, n)$*

$$\mathrm{perm}_{\,m} B \ge \frac{(sn)!\binom{n}{m}}{s^{n-m}(n-m)!((s-1)n+m)!} \frac{(r+s)!}{(r+s)^{r+s}} (\frac{r+s-1}{r+s})^{(r+s-1)(n-r-s)}. \tag{5.4}$$

Proof. Let $p = p_{m,B}(\mathbf{x})$ as defined by (2.1). Then (5.4) follow from Corollary 5.3. $\square$

Let $G_n \in \Gamma(n, r)$. Then G_n is represented by its incidence matrix $A_n \in \Delta(n, r)$. Let $B_n := \frac{1}{r} A_n$. Then B_n is a doubly stochastic matrix where each row and column of B_n has at most r positive entries. Clearly, the ALMC conjecture follows from the following stronger conjecture:

Conjecture 5.5 (The Asymptotic Lower r-Permanent Conjecture) *For $r \ge 2$, let $B_n, n = 1, 2, \ldots$ be a sequence of $n \times n$ doubly stochastic matrices, where each column of each B_n has at most r-nonzero entries. Let $k_n \in [0, n]$, $n = 1, 2, \ldots$ be a sequence of integers with $lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$. Then*

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{\,k_n} B_n}{2n} \ge gh_r(p) - \frac{p}{2} \log r. \tag{5.5}$$

Theorem 5.1 follows from the following result:

Theorem 5.6 *Let $r \ge 3, s \ge 1$ be integers. Let $B_n, n = 1, 2, \ldots$ be a sequence of $n \times n$ doubly stochastic matrices, where each column of each B_n has at most r-nonzero entries. Let $k_n \in [0, n]$, $n = 1, 2, \ldots$ be a sequence of integers with $lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$. Then*

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{\,k_n} B_n}{2n} \ge \frac{1}{2} \left(-p \log p - 2(1-p) \log(1-p)\right) + \tag{5.6}$$
$$\frac{1}{2} \left((r+s-1) \log(1 - \frac{1}{r+s}) - (s-1+p) \log(1 - \frac{1-p}{s}) \right).$$

Moreover, the Asymptotic Lower r-Permanent Conjecture 5.5 holds for $p_s = \frac{r}{r+s}, s = 0, 1, 2, \ldots$.

Proof of Theorem 5.6. Apply the inequality (5.4) to B_n for $m = k_n$. Take the logarithm of the both sides of this inequality and let $n \to \infty$. A straightforward calculation for the right-hand side, using the Stirling's formula, yields the inequality (5.6). Assume that $p = p_s = \frac{r}{r+s}$. Then $\frac{1-p_s}{s} = \frac{1}{r+s} = \frac{p_s}{r}$. Then the right-and side of (5.6) is equal to $gh_r(p_s) - \frac{p_s}{2} \log r$. Hence the asymptotic lower r-permanent conjecture 5.5 holds for $p_s = \frac{r}{r+s}, s = 1, 2, \ldots$.

We now discuss the case $s = 0$, i.e. $p = p_0 = 1$. Let $B = (b_{ij})_{i,j=1}^n$ be any $n \times n$ nonnegative matrix. Denote by $G(B) = (V, E)$ the bipartite graph induced by B, i.e. the edge (i, j) is in E, if and only if $b_{ij} > 0$. Then B induces the weighted graph on G, where the weight of the edge (i, j) is b_{ij}. Let $p_B(x) = x^n + \sum_{m=1}^n (-1)^m \mathrm{perm}_{\,m}(B)$. $p_B(x)$ is called the matching polynomial of the weighted graph G. Heilmann and Lieb showed in [22] that $p_B(x)$ has nonnegative roots. (See also [27].) Hence the arithmetic-geometric inequality for the

elementary symmetric polynomials of the nonnegative roots of $p_B(x)$ yields the inequality perm ${}_mB \ge \binom{n}{m}(\mathrm{perm}\, B)^{\frac{m}{n}}$. (See [34] for the case of m-matchings in bipartite graphs.)

Use Theorem 3.3 to deduce that perm $B_n \ge \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{r-1}{r}\right)^{(r-1)n}$. Apply the above two inequalities for the sequence B_n and $m = k_n$ for $n = 1, 2, \ldots$ to deduce the case $p = 1$. $\square$

Let $d = 6$ and $p_3 := \frac{6}{9} = \frac{2}{3}$. Then Theorem 5.1 yields that $h_3(\frac{2}{3}) \ge .7845241927$, which implies that $h_3 = \max_{p\in[0,1]} h_3(p) \ge .7845241927$. This improves the lower bound implied by (4.8) $h_3 \ge .7652789557$ [13]. The computations in [13] yield that $h_3 \le .7862023450$. Thus $h_3 \in [.7845, .7863]$.

6 Matching in general graphs - hafnians

Let $G = (V, E)$ be a graph on the set of vertices V and the set of edges E. Assume that $\#V = N$. Then G is represented by a symmetric $0-1$ matrix $B = B(G)$ with 0 diagonal. If G has a perfect matching then $N = 2n$ is even. If G is bipartite and $V = V_1 \cup V_2$, where $V_1 = \{1, \ldots, n\}, V_2 = \{n+1, \ldots, 2n\}$, we deduce that

$$B = \left(\begin{array}{cc} \mathbf{0} & A \\ A^\top & \mathbf{0} \end{array}\right), \tag{6.1}$$

where A is the representation matrix of the bipartite graph G. As explained above, the number of m-matching in the bipartite graph G is perm ${}_mA$.

In this section we discuss the m-matching of a general graph G, and the related function haf ${}_mB$ which counts the number of m-matching in G. Let $\alpha, \beta \subseteq \{1, 2, \ldots, N\}$ be two nonempty sets of cardinality i, j, $\#\alpha = i, \#\beta = j$, respectively. We then arrange the elements of $\alpha = \{\alpha_1, \ldots, \alpha_i\}$ and $\beta = \{\beta_1, \ldots, \beta_j\}$ in an increasing order: $1 \le \alpha_1 < \ldots < \alpha_i \le N, 1 \le \beta_1 < \ldots < \beta_j \le N$. For $B = (b_{st}) \in \mathbb{C}^{N\times N}$ we denote by $B[\alpha|\beta] \in C^{i\times j}$ the submatrix $(b_{\alpha_s\beta_t})_{s,t=1}^{i,j}$. Denote by $\mathrm{S}_l(\mathbb{R}) \supset \mathrm{S}_l(\mathbb{R}_+), \mathrm{S}_{l,0}(\mathbb{R}) \supset \mathrm{S}_{l,0}(\mathbb{R}_+)$ the space of real valued $l \times l$ symmetric matrices, the cone $l \times l$ symmetric matrices with nonnegative entries, the subspace of real valued $l \times l$ symmetric matrices with zero diagonal, and the subcone of $l \times l$ symmetric matrices with zero diagonal and nonnegative entries respectively. Let $B \in \mathrm{S}_N(\mathbb{R})$ and an integer $m \in [1, \frac{N}{2}]$. Then the $m-th$ *hafnian* of B is defined as

$$\mathrm{haf}\,{}_mB = 2^{-m} \sum_{\alpha,\beta\subset\{1,\ldots,N\},\#\alpha=\#\beta=m,\alpha\cap\beta=\emptyset} \mathrm{perm}\, B[\alpha, \beta]. \tag{6.2}$$

That is if $(i_1, j_1), \ldots, (i_m, j_m)$ is an m matching of a complete graph K_N on N vertices, then the product $b_{i_1j_1} \ldots b_{i_mj_m}$ appears exactly once in haf ${}_mB$. Since $b_{i_lj_l} = b_{j_li_l}$ there are 2^m choices of α and β for which this product appear, we need to use the factor 2^{-m} in the above definition of haf ${}_mB$. If $B = B(G)$ then haf ${}_mB$ gives the number of m-matching in G. Note that from the definition of perm ${}_mB$ it follows that haf ${}_mB \le 2^{-m}$perm ${}_mB$. Equivalently, it is straightforward to show:

$$\mathrm{haf}\,{}_mB = (2^m m!)^{-1} \sum_{1\le i_1<\ldots<i_{2m}\le N} \frac{\partial^{2m}}{\partial x_{i_1}\ldots\partial x_{i_N}} (\mathbf{x}^\top B\mathbf{x})^m, \quad B \in \mathrm{S}_N(\mathbb{R}). \tag{6.3}$$

Unfortunately, the quadratic polynomial $\mathbf{x}^\top B\mathbf{x}$ is not always positive hyperbolic. Note that haf ${}_mB$ does not depend on the value of the diagonal entries B. Let $B^{(0)}$ be the matrix obtained from by replacing the diagonal entries of B by zero elements. Then haf ${}_mB =$ haf ${}_mB^{(0)}$.

For $B \in \mathrm{S}_l(\mathbb{R})$ we denote by $\lambda_1(B) \ge \ldots \ge \lambda_l(B)$ the l eigenvalues of B counted with their multiplicities and arranged in the decreasing order. As usual for $B, C \in \mathrm{S}_l(\mathbb{R})$ we denote $C \succeq B$ if $C - B$ is a nonnegative definite matrix. The maxmin, or minmax charaterization of $\lambda_k(B)$ yields that if $C \succeq B$ then $\lambda_k(C) \ge \lambda_k(B)$ for $k = 1, \ldots, l$. In

particular, if $B \in \mathrm{S}_l(\mathbb{R})$ has nonnegative diagonal entries then $B \succeq B^{(0)}$ and $\lambda_k(B) \geq \lambda_k(B^{(0)})$ for $k = 1, \ldots, l$.

The following result is well known and we bring its proof for completeness.

Lemma 6.1 *Let $B \in \mathrm{S}_n(\mathbb{R})$ and $n \geq 2$. Then $\mathbf{x}^\top B\mathbf{x}$ is positive hyperbolic if and only if $\mathbf{0} \neq B \in \mathrm{S}_n(\mathbb{R}_+)$ and $0 \geq \lambda_2(B)$.*

Proof. Assume that $\mathbf{x}^\top B\mathbf{x}$ is positive hyperbolic. Then Proposition 2.2 yields that $B \in \mathrm{S}_n(\mathbb{R}_+)$. Since $\mathbf{x}^\top B\mathbf{x} > 0$ for each $\mathbf{x} > \mathbf{0}$ $B \neq \mathbf{0}$. Hence $\lambda_1(B) > 0$.

Observe next that the positive hyperbolicity of $\mathbf{x}^\top B\mathbf{x}$ is equivalent to

$$(\mathbf{x}^\top B\mathbf{y})^2 \geq (\mathbf{x}^\top B\mathbf{x})(\mathbf{y}^\top B\mathbf{y}), \text{ for any } \mathbf{x} > \mathbf{0}, \mathbf{y} \in \mathbb{R}^n. \tag{6.4}$$

Clearly, the above condition holds for any $\mathbf{x} \geq \mathbf{0}$. The Perron-Frobenius theorem yields that there exists $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n_+$ such that $B\mathbf{x} = \lambda_1(B)\mathbf{x}$. Let $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^n$ and assume that $\mathbf{y}^\top \mathbf{x} = 0$. Then (6.4) yields that $\mathbf{y}^\top B\mathbf{y} \leq 0$. Hence $\lambda_2(B) \leq 0$.

Vice versa suppose that $\mathbf{0} \neq B \in \mathrm{S}_n(\mathbb{R}_+)$ and $\lambda_2(B) \leq 0$. Recall that there exists a permutation matrix $P \in \{0,1\}^{n\times n}$ such that $P^\top BP$ is a block diagonal matrix $\mathrm{diag}(B_1, \ldots, B_k)$, where $B_i \in \mathrm{S}_{n_i}(\mathbb{R}_+)$ is irreducible. So $\lambda_1(B_i) > 0$ unless $n_i = 1$ and $B_i = 0$. Hence our assumptions yield that we may assume that $\mathbf{0} \neq B_1 \in \mathrm{S}_{n_1}(\mathbb{R}_+)$ is irreducible and $B_i = 0$ for $i = 2, \ldots, k$. Thus it is enough to show that $\mathbf{x}^\top B\mathbf{x}$ is positive hyperbolic for an irreducible $B \in \mathrm{S}_n(\mathbb{R}_+)$, where $n \geq 2$. Clearly $\mathbf{x}^\top B\mathbf{x} > 0$ for $\mathbf{x} > 0$. Thus it is left to show that (6.4) holds. Assume first that $B\mathbf{x} = \lambda_1(B)\mathbf{x}, \mathbf{x} > 0$. Then (6.4) follows straightforward. Suppose $\mathbf{x} > 0$ is any vector. Then there exists a unique diagonal matrix D, with positive diagonal entries such that $B\mathbf{x} = D^{-2}\mathbf{x}$. That is $DBD(D^{-1}\mathbf{x}) = D^{-1}\mathbf{x}$. Replacing $B, \mathbf{x}, \mathbf{y}$ by $DBD, D^{-1}\mathbf{x}, D^{-1}\mathbf{y}$ we deduce the inequality second inequality of (6.4). $\square$

Definition 6.2 *Let $G = (V, E)$ be a graph on the set of vertices V. Then for $k \geq 2$ G is called k-partite, if the following condition holds. There exists a decomposition of V to a disjoint union of k nonempty sets $V_1, \ldots, V_k$ such that each edge $e \in E$ connects V_i to V_j for some $i \neq j$. G is called a complete k-bipartite, if there exists a decomposition of V to a disjoint union of k nonempty sets $V_1, \ldots, V_k$ such that E consists of all edges from V_i to V_j for all $1 \leq i < j \leq k$.*

Note that a complete graph G with $n \geq 2$ vertices is complete n- partite.
We need the following elementary result whose proof is straightforward.

Proposition 6.3 *Let $n \geq 3$ and $F = [f_{ij}] \in \mathrm{S}_n(\mathbb{R})$ with $1's$ on the main diagonal. Suppose that and all subsets of cardinality three $\alpha = \{1 \leq i < j < k \leq n\}$ the principal submatrix $F[\alpha, \alpha]$ is nonnegative definite . If $f_{ij} = 1, f_{jk} = 1$ then $f_{ik} = 1$.*

Lemma 6.4 *Let $B = [b_{ij}] \in \mathrm{S}_n(\mathbb{R}_+)$, i.e. B is a real $n \times n$ symmetric matrix with nonnegative entries. Denote by $G(B) = (\langle n\rangle, E)$ the graph, (with no self-loops), induced by B, i.e. $(i, j) \in E$ if and only if $i \neq j$ and $b_{ij} > 0$. Assume that B is irreducible and $\lambda_2(B) \leq 0$. Then $G(B)$ is a complete k-partite graph for some $k \in [2, n]$.*

Proof. Since $B^{(0)}$ is irreducible and $B \succeq B^{(0)}$, it is enough to prove the lemma in the case where all the diagonal entries of B are equal to zero, i.e. $B = B^{(0)}$. Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ be the unique positive eigenvector of B, corresponding to the maximal eigenvalue $\lambda_1(B)$, satisfying the condition $\mathbf{x}^\top \mathbf{x} = \lambda_1(B)$. Then $A = \mathbf{x}\mathbf{x}^T - B$ and $A \succeq \mathbf{0}$. Let $D = \mathrm{diag}(x_1, ..., x_n)$. Then the zero pattern of the matrix $C = D^{-1}BD^{-1}$ is the same as of the matrix B. Let $F = [f_{ij}] := \mathbf{1}\mathbf{1}^\top - C$, where $\mathbf{1} := (1, \ldots, 1)^\top$ is the vector of all ones. Then $F = [f_{ij}] \succeq \mathbf{0}$. Notice that $f_{ii} = 1, 1 \leq i \leq n$, and for $i \neq j$ $f_{ij} = 1$ if and only $b_{ij} = 0$, (i.e. the vertices i, j are not connected in the graph $G(B)$). As any principal submatrix of F is nonnegative definite, it follows from Proposition 6.3 that for any triplet

$1 \le i < j < k \le n$ such that if $f_{ij} = 1, f_{jk} = 1$ then also $f_{ik} = 1$. In other words, the relation "$i \sim j$" $\iff$ "$b_{ij} = 0$" is an equivalence relation. Therefore the graph $G(B)$ is complete k-partite, where each equivalence class of vertices corresponds to some class V_i in the k-partite graph.

Theorem 6.5 *Let G be a graph on $n > 1$ vertices, and denote by $A(G)$ the incidence matrix of G. Then $\mathbf{x}^\top A(G)\mathbf{x}$ is positive hyperbolic if and only if G is a union of a complete $k(\ge 2)$-partite graph on at least two vertices and of isolated vertices.*

Proof. We first show that for a complete k-partite graph G on at least $n \ge 2$ vertices $\lambda_2(G) \le 0$, which is equivalent to the positive hyperbolicity of $\mathbf{x}^\top A(G)\mathbf{x}$ in view of Lemma 6.1. Let J_n be a symmetric matrix whose all entries are equal to 1. Then J is rank one matrix with $\lambda_1(J_n) = n$ and $\lambda_i(J_n) = 0$ for $i = 2, \ldots, n$. Let $k \in [1, n]$, $n_i \in \mathbb{N}, i = 1, \ldots, k, 1 \le n_k \le \ldots \le n_1$ and $n_1 + \ldots + n_k = n$. Consider the block diagonal matrix $J(n_1, \ldots, n_k) := \mathrm{diag}(J_{n_1}, \ldots, J_{n_k})$. Clearly $J(n_1, \ldots, n_k)$ is a nonnegative definite matrix. It is straightforward to see that renaming the vertices of G, we will obtain that $A(G) = J_n - J(n_1, \ldots, n_k)$ for some unique $n_1 \ge \ldots n_k \ge 1$. Then minimax characterization of $\lambda_2(A(G))$ yields that $\lambda_2(A(G)) \le \lambda_2(J_n) = 0$. Hence $\mathbf{x}^\top A(G)\mathbf{x}$ is positive hyperbolic.

Assume now that $\mathbf{x}^\top A(G)\mathbf{x}$ is positive hyperbolic. Therefore G must have at least one edge and $\lambda_2(A(G)) \le 0$. Hence G has one connected component containing at least two vertices and a union of isolated vertices. Without loss of generality we assume that G is connected. Then $A(G)$ satisfies the assumptions of Lemma 6.4 and G is k-partite. $\square$

Remark 6.6 *Let A be a symmetric $n \times n$ matrix with nonnegative entries. It is straightforward to show that the polynomial $\mathbf{x}^\top A\mathbf{x}$ is positive hyperbolic if and only if the function $\sqrt{\mathbf{x}^\top A\mathbf{x}}$ is concave on the positive orthant R^n_+. If A is real, symmetric, nonnegative definite then $\sqrt{\mathbf{x}^\top A\mathbf{x}}$ is convex on R^n. In view of Remark 2.9 , it is natural to conjecture that if A is $2n \times 2n$ real, symmetric, nonnegative definite then the reverse van der Waerden bound holds :*

$$2^n n! \mathrm{haf}\,_n B \le \frac{(2n)!}{(2n)^{2n}} \mathrm{Cap}\,(p), \quad p(\mathbf{x}) := (\mathbf{x}^\top A\mathbf{x})^n.$$

7 Algorithmic applications

One of the main purposes of this paper is to construct a generating homogeneous polynomial $p(\mathbf{x})$ of degree n, for some quantity of interest Q as hafnian: haf A, sums of subhafnians: haf $_m A$, permanent: perm A, sum of subpermanents: $\mathrm{perm}_m\, A$, such that $Q = \frac{\partial^n p}{\partial x_1 \ldots \partial x_n}(\mathbf{0})$. If such polynomial is positive hyperbolic then we can apply the results from [19] and the results of the previous sections of this paper to get a lower bound on Q, and even to get deterministic polynomial-time algorithms to approximate Q within simply exponential factor as in [19]. In the general, (not positive hyperbolic case), we can use this representation to obtain exact algorithm to compute Q in $2^n poly(n)$ number of arithmetic operations provided the the generating polynomial p can be itself evaluated in $poly(n)$ number of arithmetic operations. We present below some examples of this approach.

7.1 Formula for $\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0})$

Our exact algorithms are based on the following elementary identity 7.2.

Let $p(\mathbf{x})$ be a polynomial of degree m in $n \ge m$ variables , $p(0) = 0$. Define

$$s_i = \sum_{b_j \in \{0,1\}, \sum 1 \le j \le m = i} p(b_1, ..., b_m). \tag{7.1}$$

Let $\mathbf{d}_n = (d_{n,1}, ..., d_{n,m-1})$ be the unique solution of the system of linear equations $\mathbf{d}_n A = (-1, ..., -1)$, where the $m-1 \times m-1$ lower triangular matrix $A = [a_{ij}]$ is defined as follows:

$$a_{ij} = \binom{n-j}{i-j} \text{ if } i \geq j \text{ and } a_{ij} = 0 \text{ otherwise.}$$

Then the following equality holds :

$$\sum_{1 \leq i_1 < ... < i_m \leq n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) = p(1, ..., 1) + \sum_{1 \leq j \leq m-1} s_j d_{n,j} \tag{7.2}$$

Notice that this formula requires $\sum_{0 \leq j \leq n-1} \binom{n}{j}$ evaluations of the polynomial p.

The formula 7.2 follows from the following obvious identities :

$$s_i = \sum_{1 \leq j \leq i} \binom{n-j}{i-j} c_j, 1 \leq i \leq m,$$

where c_j is the sum of the coefficients of all monomials in p involving exactly j variables.

The formula 7.2 is, in a sense, optimal for $n = m$, i.e., if for all homogeneous polynomials $p(\mathbf{x})$ of degree n

$$\frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{x}) = \sum_{1 \leq i \leq k} a_i p(\mathbf{z}_i), \quad a_i \in \mathbb{C}, \mathbf{z}_i \in \mathbb{C}^n,$$

then $k \geq \binom{n}{\frac{n}{2}} \approx \frac{2^n}{\sqrt{n}}$ [18].

7.2 Ryser' like formulas for sums of subhafnians and subpermanents

1. Let $B \in \mathrm{S}_N(\mathbb{R}), \mathbf{x} := (x_1, \ldots, x_N)^\top \in \mathbb{C}^N$ and $m \in [1, n] \cap \mathbb{N}$. Define, as in the proof of Theorem 3.1, the polynomial :

$$P_m(\mathbf{x}) = \frac{1}{2^m m!}(\mathbf{x}^\top B \mathbf{x})^m$$

As $\mathrm{haf}\ _m B = (2^m m!)^{-1} \sum_{1 \leq i_1 < ... < i_{2m} \leq N} \frac{\partial^{2m}}{\partial x_{i_1} ... \partial x_{i_{2m}}}(\mathbf{x}^\top B \mathbf{x})^m$, the application of 7.2 gives the following Ryser-like formula for $\mathrm{haf}\ _m B$:

$$\mathrm{haf}\ _m B = P_m(1, ..., 1) + \sum_{1 \leq j \leq m-1} s_j d_{n,j}, \tag{7.3}$$

where s_i are defined by (7.1) for $p = P_m$. The formula (7.3) provides $SB(N, m)(O(N^2) + O(\log(m)))$, algorithm to compute $\mathrm{haf}\ _m B$, where $SB(N, m) := \sum_{0 \leq j \leq 2m-1} \binom{N}{j}$.

2. For $\mathbf{x} := (x_1, ..., x_n)^\top \in \mathbb{C}^n$ let

$$S_m(\mathbf{x}) = \sum_{1 \leq i_1 < ... < i_m \leq n} x_{i_1} ... x_{i_m}.$$

be the $m - th$ symmetric function of $\mathbf{x}$. Let A be $n \times n$ complex matrix and define $p_m(\mathbf{x}) := S_m(A\mathbf{x})$. Then $p_m(\mathbf{x})$ can be evaluated in $O(n^2)$ arithmetic operations and

$$\mathrm{perm}_m A = \sum_{1 \leq i_1 < ... < i_m \leq n} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} p_m(\mathbf{x}).$$

Which gives the following algorithm to evaluate $\mathrm{perm}_m A$

$$\mathrm{perm}_m A = p_m(1, ..., 1) + \sum_{1 \leq j \leq m-1} s_j d_{n,j}, \tag{7.4}$$

where s_i are defined by (7.1) for $p = p_m$. The formula 7.4 provides $SB(N,m)(O(N^2))$ algorithm to compute $\mathrm{perm}_m A$.

Notice that the naive algorithm , i.e. computing and adding all $m \times m$ subpermanents, requires $(\binom{n}{m})^2 2^m O(m)$ arithmetic operations .

7.3 Positive hyperbolic polynomials and convex relaxations

In this section we always assume that $\mathbf{x} = (x_1, \dots, x_n)^\top, \mathbf{y} = (y_1, \dots, y_n)^\top \in \mathbb{C}^n$, $\mathbf{1} = (1, \dots, 1)^\top$. Suppose that a positive hyperbolic polynomial

$$p(\mathbf{x}) = \sum_{\sum_{1 \le i \le n} r_i = n} a_{(r_1,\dots,r_n)} \prod_{1 \le i \le m} x_i^{r_i}, m \ge n$$

has nonnegative integer coefficients and is given as an oracle. I.e. we don't have a list coefficients , but can evaluate $p(\mathbf{x})$ on rational inputs . The number $\log p(\mathbf{1})$ measures the complexity of the polynomial p .

A deterministic polynomial-time oracle algorithm is any algorithm which evaluates the given polynomial $p(\mathbf{x})$ at a number of rational vectors $\mathbf{q}^{(i)} = (q_1^{(i)}, ..., q_n^{(i)})$ which is polynomial in n and $\log p(\mathbf{1})$; these rational vectors $\mathbf{q}^{(i)}$ are required to have bit-wise complexity which is polynomial in n and $\log p(\mathbf{1})$; and the number of additional auxiliary arithmetic computations is also polynomial in n and $\log p(\mathbf{1})$.
If the number of oracle calls, (evaluations of the given polynomial $p(\mathbf{x})$), the number of additional auxiliary arithmetic computations and bit-wise complexity of the rational input vectors $\mathbf{q}^{(i)}$ are all polynomial in n, (no dependence on $\log p(\mathbf{1})$) then such algorithm is called deterministic strongly polynomial-time oracle algorithm.
The following theorem was proved in [19].

Theorem 7.1 *There exists a deterministic polynomial-time oracle algorithm, which computes for given as an oracle indecomposable positive hyperbolic polynomial $p(\mathbf{x})$ a number $F(p)$, satisfying the inequality*

$$\frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0}) \le F(p) \le 2 \frac{n^n}{n!} \frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0})$$

Our goal in this paper is to extend Theorem 7.1 to approximate $\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m q}{\partial x_{i_1} ... \partial x_{i_m}}(\mathbf{0})$ for a given positive hyperbolic polynomial $q(\mathbf{x})$ of degree m .

The algorithm behind Theorem 7.1 is based on two observations . First,

$$\log \mathrm{Cap}\, p = \inf_{\sum_{1 \le i \le n} y_I = 0} \log p(e^{\mathbf{y}}), \tag{7.5}$$

where we used the notation of Proposition 2.3. If the coefficients of the polynomial p are nonnegative then the functional $\log p(e^{\mathbf{y}})$ is convex , the indecomposabilty of the polynomial p is exactly uniqueness and existence of the minimum in 7.5 .
Second point is the inequality :

$$\mathrm{Cap}\, p \frac{n!}{n^n} \le \frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0}) \le \mathrm{Cap}\, p \tag{7.6}$$

If the positive hyperbolic polynomial p is not indecomposable , we need first to check if $\mathrm{Cap}\, p > 0$. If $\mathrm{Cap}\, p = 0$ then also $\frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0}) = 0$. In the case $\mathrm{Cap}\, p > 0$ we "slightly" perturb the polynomial p to get the indecomposability.
Theorem 3.1 in this paper provides an analogue of the left inequality in (7.6) for positive hyperbolic polynomial $q(\mathbf{x})$ of degree $m < n$. The problem is that in this case we don't have the right inequality. I.e. it is possible that $\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m}{\partial x_{i_1} ... \partial x_{i_m}} q(\mathbf{0}) > 0$ but $\mathrm{Cap}\,(q) = 0$. This problem can be easily overcome by the following equivalent reformulation of Theorem 3.1:

Theorem 7.2 *Consider a positive hyperbolic polynomial $q(\mathbf{x})$ of degree $m < n$. Define a positive hyperbolic polynomial $p(\mathbf{x}) = q(\mathbf{x})(\frac{\sum_{1 \le i \le n} x_i}{n})^{n-m}$ and*

$$D_m(q) = \frac{(n-m)!}{n^{n-m}} \sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m q}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}).$$

Then the following inequality holds :

$$\frac{n!}{n^n} \mathrm{Cap}\ p \le D_m(q) = \frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0}) \le \mathrm{Cap}\ p. \tag{7.7}$$

Corollary 7.3 *Let $q(\mathbf{x})$ be a positive hyperbolic polynomial of degree $m < n$ given as an oracle. Then there exists a strongly polynomial-time, (in n), oracle algorithm to check if $\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m q}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) > 0$.*

Proof. It follows from Theorem 7.2 that $\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m q}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) > 0$ if and only if $\frac{\partial^n p}{\partial x_1 ... \partial x_n}(\mathbf{0}) > 0$, where $p(\mathbf{x}) = q(\mathbf{x})(\frac{\sum_{1 \le i \le n} x_i}{n})^{n-m}$ is positive hyperbolic polynomial of degree n . Notice the polynomial q is easy to evaluate given an oracle evaluating the polynomial p . Let $q(\mathbf{x}) = \sum_{\sum_{1 \le i \le n} r_i = n} a_{r_1,...,r_n} x_1^{r_1} ... x_n^{r_n}$, the support $Supp(q) = \{(r_1, ..., r_n) : a_{r_1,...,r_n} \neq 0$, the Newton polytope is the convex hull of the support $CO(Supp(q))$. It was proved in [18] that an integer vector $(r_1, ..., r_n) \in Supp(q)$ if and only if $(r_1, ..., r_n) \in CO(Supp(q))$. Corollary 4.3 in [19] provides a strongly polynomial (in n) oracle algorithm for the membership problem "$X \in CO(Supp(q))$?" for positive hyperbolic polynomial $q(\mathbf{x})$ of degree n .

□

Corollary 7.4 *Let $q(\mathbf{x})$ be a positive hyperbolic polynomial of degree $m < n$ given as an oracle. Then*

1. *There exists a strongly polynomial-time, (in n), oracle algorithm to check if* $D_m(q) := \sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m q}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) > 0$.
2. *There exists a deterministic polynomial-time oracle algorithm which computes a number $F_m(q)$ satisfying the inequality $1 \le \frac{F_m(q)}{D_m(q)} \le 2\gamma(n,m) \le 2\frac{n^n}{n!}$.*

Corollary 7.5 *Let A be $n \times n$ matrix with nonnegative entries. Then there exists a deterministic polynomial-time algorithm which computes a number $P_m(A)$ satisfying the inequality $1 \le \frac{P_m(A)}{\mathrm{perm}_m A} \le 2\gamma(n,m) \le 2\frac{n^n}{n!}$.*

It s very possible that there exists a deterministic polynomial-time algorithm (in n) which approximates $\mathrm{perm}_m A$ within multiplicative factor e^m .

7.4 A conjecture

Let the assumptions of Corollary 7.4 holds. To approximate $D_m(q)$ we used the identity

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(\mathbf{0}) r(\mathbf{0}) = D_q(m), \tag{7.8}$$

for $r(\mathbf{x}) = ((n-m)!)^{-1}(\sum_{i=1}^n x_i)^{n-m}$. It is natural to ask we can improve our estimates if we choose a different positive hyperbolic $r(\mathbf{x})$ such that (7.8) holds.

Definition 7.6 *Denote the set of positive hyperbolic polynomials of degree m and in n variables as $PHYP(n,m)$; and by $UPHYP(n,m) \subset PHYP(n,m)$ the subset of polynomials satisfying $\frac{\partial^{n-m}}{\partial x_{i_1}\ldots\partial x_{i_{n-m}}} r(\mathbf{0}) = 1$ for all $1 \le i_1 < \ldots < i_{n-m} \le n$. Define*

$$\gamma(n,m) = \inf_{r \in UPHYP(n,n-m)} \sup_{q \in PHYP(n,m)} \frac{\mathrm{Cap}\ qr}{\frac{\partial^n}{\partial x_1 \ldots \partial x_n} q(0,...0) r(0,...,0)}.$$

Assume that $r \in UPHYP(n,n-m)$. Since r is positive hyperbolic, all its monomials have nonnegative coefficients. Hence $r(\mathbf{x}) \ge S_{n,n-m}(\mathbf{x})$ for any $\mathbf{x} \ge 0$. In particular $\mathrm{Cap}\ r \ge \mathrm{Cap}\ S_{n,n-m} = \binom{n}{m}$. By choosing in inf sup definition $\gamma(n,m)$ $q = (\frac{x_1+\ldots+x_n}{n})^m$ we deduce straightforward that $\gamma(n,m) \ge \frac{n^m}{m!}$.

Conjecture 7.7

$$\gamma(n,m) = \sup_{q \in PHYP(n,m)} \frac{\mathrm{Cap}\ qS_{n,n-m}}{\frac{\partial^n}{\partial x_1 \ldots \partial x_n} q(\mathbf{0}) S_{n,n-m}(\mathbf{0})} = \frac{n^m}{m!}.$$

Note that the hyperbolic van der Waerden inequality [19] implies that $\gamma(n,m) \le \frac{n^n}{n!}$. It follows from Theorem 3.1 that for each $q \in PHYP(n,m)$ such that $\mathrm{Cap}\ q = q(\mathbf{1})$ we have the inequality

$$\frac{\mathrm{Cap}\ qS_{n,n-m}}{\frac{\partial^n}{\partial x_1 \ldots \partial x_n} q(\mathbf{0}) S_{n,n-m}(\mathbf{0})} \le \frac{n^m}{m!}.$$

Remark 7.8 *We presented in this section one "natural" generating polynomial for the hafnian, and described all symmetric boolean matrices such that this polynomial is positive hyperbolic. It is an interesting open problem whether there exists a generating positive hyperbolic polynomial for the hafnians of boolean matrices which can can evaluated in polynomial time. If the answer to this problem is negative it can explain why approximating the hafnian (number of perfect matchings in general graphs) is "harder" than the same problem for the permanent. It is easy to prove that computing the hafnian of integer symmetric $2n \times 2n$ matrices with nonnegative entries and the signature $(+,-,...,-)$ if $\#P$-complete. The results in this paper allow to use Sinkhorn's scaling to approximate the hafnian within multiplicative factor e^{2n} in this "hyperbolic" case.*

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